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NEW THETA CONSTANT IDENTITIES

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Dedicated to John Thompson for his many original contributions to mathematics

ABSTRACT

The residue theorem is employed to obtain new identities among pth powers of theta constants with rational characteristics. The technique is then used to derive some known identities of Ramanujan.

Introduction

The subject of theta constant identities is an old one. Classically, the identities were among theta constants with integer characteristics. The automorphic forms for the modular group were constructed as homogeneous polynomials in these theta constants and suitable powers of these theta constants were easily seen to be automorphic forms for the principal congruence subgroup of the modular group of level 2. In addition, quotients of the 4th powers of these constants gave the classical λ function, namely the holomorphic mapping of the upper half plane onto the thrice punctured sphere. For the theory of theta identities and background the reader is referred to [RF], [FK1], [I] and [M]. Recently a method was discussed in [BB1, BB2] which uses the arithmetic geometric mean and goes back to Gauss.

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More recently [FK1], theta functions with rational characteristics were studied and were used to construct modular forms for the principal congruence subgroups of the modular group of level p and examples of holomorphic mappings of the upper half plane onto a four times and twelve times punctured sphere were constructed. In addition, two new theta identities were given related to the case $p=3$.

It is the purpose of this note to give what seems to be a new method of obtaining theta identities for the one dimensional theta constants which generalize the aforementioned ones. The technique we use to derive the identities is classical. We compute the dimension of certain spaces of functions on a torus, and use this to construct a singular matrix. An analysis of this matrix then suggests the identities which are then proved via the residue theorem.

I. Definitions and Generalities

We begin with the definition of the one dimensional theta function and its properties:

Definition: For $\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right] \in \mathbb{R}^2$, $\tau \in \mathbb{H}$, $z \in \mathbb{C}$, we define:

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau)=\sum_{n\in\mathbf{Z}}\exp 2\pi i\left\{(1/2)\left(n+\frac{\epsilon}{2}\right)^2\tau+\left(n+\frac{\epsilon}{2}\right)\left(z+\frac{\epsilon'}{2}\right)\right\}.$$

This series is uniformly and absolutely convergent on compact subsets of $\mathbb{C}\times\mathbb{H}$. The main properties of theta functions that we shall require are: For m,n integers,

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z+n+m\tau,\tau)=\exp 2\pi i\left\{\frac{n\epsilon-m\epsilon'}{2}-mz-\frac{m^2}{2}\tau\right\}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau).$$

For m, n arbitrary real numbers

$$\begin{aligned}(1)\quad &\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z+\tau\frac{m}{2}+\frac{n}{2},\tau)\\ &\qquad=\exp 2\pi i\left\{-\frac{1}{2}mz-\frac{1}{8}m^2\tau-\frac{1}{4}m(\epsilon'+n)\right\}\theta\begin{bmatrix}\epsilon+m\\ \epsilon'+n\end{bmatrix}(z,\tau).\end{aligned}$$

In addition we have for m,n integers

$$\theta\begin{bmatrix}\epsilon+2m\\ \epsilon'+2n\end{bmatrix}(z,\tau)=\exp\pi i\{\epsilon n\}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau)\tag{2}$$

and finally

$$\theta\begin{bmatrix} -\epsilon \\ -\epsilon' \end{bmatrix}(z,\tau) = \theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(-z,\tau). \tag{3}$$

It is a well known fact that the one dimensional theta function has precisely one zero in the fundamental period parallelogram and that the zero of $\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](z,\tau)$ is at the point $\frac{1-\epsilon}{2}\tau + \frac{1-\epsilon'}{2}$. If we assume that $\epsilon = \frac{m}{p}$, $\epsilon' = \frac{n}{p}$ with m,n odd integers between 1 and $2p-1$ we see that the zeros of these functions are points of order p in the associated torus. We shall denote the zero of $\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](z,\tau)$ by $P_{\epsilon,\epsilon'} = \frac{1-\epsilon}{2}\tau + \frac{1-\epsilon'}{2}$. It is an elementary consequence of the Riemann Roch theorem that on a torus the dimension of the space of meromorphic functions with a pole of order n at a point P and regular elsewhere is n. Furthermore, any quotient of pth powers

$$\frac{\theta^p\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(z,\tau)}{\theta^p\begin{bmatrix} \delta \\ \delta' \end{bmatrix}(z,\tau)}$$

is a meromorphic function on the torus provided both characteristics are of the form $\frac{m}{P}, \frac{n}{P}$ with m,n odd. It therefore follows that if we consider the $p+1$ characteristics

$$\left\{\begin{bmatrix} \frac{1}{p} \\ \frac{1}{p} \end{bmatrix}, \ldots, \begin{bmatrix} \frac{1}{p} \\ \frac{2p-1}{p} \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{p} \end{bmatrix}\right\},$$

the associated pth powers of theta functions must satisfy a linear relation. This is our point of departure: There exist constants independent of z but may depend on τ, such that

$$\sum_{n=1}^{n=p} c_n\theta^p\begin{bmatrix} \frac{1}{p} \\ \frac{2n-1}{p} \end{bmatrix}(z,\tau) + c_{n+1}\theta^p\begin{bmatrix} 1 \\ \frac{1}{p} \end{bmatrix}(z,\tau) = 0.$$

The linear dependence above guarantees the existence of a non-trivial vector $(c_1, \ldots, c_{n+1})$ for which the above is true. In particular, therefore, we can substitute various values of z and obtain a set of homogeneous linear equations and be guaranteed a non-trivial solution.

The idea now is to insert for z the zeros of the ordered set of functions $\theta\left[\begin{smallmatrix}\frac{1}{p}\\ \frac{2l-1}{p}\end{smallmatrix}\right](z,\tau)$ and the zero of the function $\theta\left[\begin{smallmatrix}1\\ \frac{1}{p}\end{smallmatrix}\right](z,\tau)$ and in this way to obtain a $(p+1,p+1)$ matrix which is singular. When you do this for $p=3$ and set the determinant equal to zero you recover the identity derived in [FK1]. The observation we make is that the determinant of the p by p matrix in the upper left hand corner is in fact equal to zero and we can find a vector in $\mathbb{C}^p$ which annihilates the p columns of this matrix. The theta identities we derive say that this vector also annihilates the last column.

II. Main Results

The theorem we wish to prove is the following:

THEOREM:

$$\sum_{l=1}^{l=p}(-1)^{l+1}\theta^p\begin{bmatrix}\frac{1}{p}\\ \frac{2l-1}{p}\end{bmatrix}(0,\tau)=0.$$

This theorem will follow from the following result which we call

THEOREM 1: *Let f be a holomorphic function on the plane which satisfies the following functional equations:*

$$f(z+1)=-f(z)$$

and

$$f(z+\tau)=-\exp 2\pi i\Big(-pz-p\frac{\tau}{2}\Big)f(z)$$

Then

$$\sum_{l=0}^{l=p-1}(-1)^l\omega^{2l+1}f\Big(\frac{p-1}{2p}\tau+\frac{l}{p}\Big)=0$$

where

$$\omega=\exp(\pi i\frac{p-1}{2p}).$$

Finally, Theorem 1 will follow from the well known fact that the sum of the residues of an elliptic function is zero.

The main observation is the following lemma:

LEMMA 1: *The function $\theta\left[\begin{smallmatrix}\frac{m}{p}\\ 1\end{smallmatrix}\right](pz,p\tau)$ satisfies the functional equations and vanishes at the p points $\{\frac{p-m}{2p}\tau+\frac{l}{p}\}$, $l=0,..,(p-1)$.*

Proof: The proof is applying the functional equation satisfied by the theta function given above and observing that the zero of the function, $\theta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](z,p\tau)$, under consideration is at the point $\frac{1+p\tau}{2}$ in the parallelogram whose sides are the line joining zero and one and the line joining zero and $p\tau$. We then note that the points in the parallelogram whose sides are the line joining zero and one and the line joining zero and τ which map to this point under multiplication by p is exactly the set of points given in the statement of the lemma.

In other words we have found that the zeros of the function in the lemma when, for example, we choose $m = 1$, lie in a fundamental region for the elliptic function with periods $1, \tau$. The conclusion therefore is that

$$\frac{f}{\theta\begin{bmatrix}\frac{1}{p}\\1\end{bmatrix}(pz, p\tau)}$$

is an elliptic function with periods $1, \tau$ with simple poles at these points provided f does not vanish at any of these points. If f does vanish at some of the points, there will not be a pole at those points. The sum of the residues of an elliptic function in a period parallelogram must vanish so we have that

$$\sum_{l=0}^{l=p-1} \operatorname{Res}_{z_l} \frac{f}{\theta\begin{bmatrix}\frac{1}{p}\\1\end{bmatrix}(pz, p\tau)} = 0$$

where

$$z_l = \frac{p-1}{2p}\tau + \frac{l}{p}.$$

The point now is that

$$\operatorname{Res}_{z_l} \frac{f}{\theta\begin{bmatrix}\frac{1}{p}\\1\end{bmatrix}(pz, p\tau)} = \frac{f(z_l)}{\theta'\begin{bmatrix}\frac{1}{p}\\1\end{bmatrix}(pz_l, p\tau)}$$

and we can compute this expression.

LEMMA 2: $\theta'\begin{bmatrix}\frac{1}{p}\\1\end{bmatrix}(pz_l, p\tau) = C(\tau)(-1)^l \exp(-\pi i \frac{p-1}{2p}(2l+1))$.

Proof: According to the properties of the theta function, in particular equation (1), we have that translation of the variable z by $\frac{r}{2}\tau + \frac{s}{2}$ multiplies the original function by the exponential of a linear function of z and translates the characteristic by the vector $\begin{bmatrix}r\\s\end{bmatrix}$. Hence the derivative of our function at the point in question can be calculated using the appropriate formula. While the result in general is a complicated expression, the result here is simplified since, after translating the characteristic, we obtain the odd integer characteristic and are evaluating the function at the origin. This proves the lemma and also finishes the proof of Theorem 1.

In order to obtain the theta identity given we simply have to apply Theorem 1 to the function

$$f = \theta^p \begin{bmatrix} 1 \\ \frac{1}{p} \end{bmatrix} (z,\tau).$$

In other words we must compute $\theta^p \left[\begin{smallmatrix} 1 \\ \frac{1}{p} \end{smallmatrix}\right](z_l,\tau)$. The same formula we used in the previous computation now gives this equal to

$$D(\tau)\exp(-\pi i \frac{p-1}{2p}(2l+1))\theta^p \begin{bmatrix} \frac{1}{p} \\ \frac{2p-2l+1}{p} \end{bmatrix}.$$

We have therefore obtained

$$\sum_{l=1}^{l=p}(-1)^l\theta^p \begin{bmatrix} \frac{1}{p} \\ \frac{2p-2l+1}{p} \end{bmatrix}(0,\tau) = 0$$

which is the identity we wanted to derive.

We point out that we can derive many other types of identities in this manner. Also, the method recaptures many known theta identities which are classical. We close this paper with some of the possible identities.

I.

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(0,\tau)\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(0,3\tau) = \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(0,\tau)\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(0,3\tau) + \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,3\tau).$$

This identity follows from the residue theorem by considering the function

$$\frac{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix}(3z,3\tau)}{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(z,\tau)\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(z,\tau)\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}(z,\tau)}.$$

This is a well known and classical identity which goes by the name **cubic modular equation**. This identity can be found in [BB2, p.110].

II. An additional identity essentially equivalent to one due to Ramanujan [B, p. 282 (xii)] is

$$\frac{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,5\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)}-\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(0,5\tau)}{\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)}-\frac{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,5\tau)}{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)}$$
$$=-5\frac{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,5\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,5\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,5\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)}.$$

This identity is also an immediate consequence of the residue theorem. You simply apply the residue theorem to the function

$$\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(5z,5\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(z,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(z,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(z,\tau)\theta^2\begin{bmatrix}1\\1\end{bmatrix}(z,\tau)}$$

and observe that this function has only a simple pole at the origin.

III. A third identity which follows also from the theory described here, but not immediately, is the so-called **septic modular equation** [BB2, p.112]:

$$\sqrt{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,7\tau)}=\sqrt{\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,7\tau)}$$
$$+\sqrt{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,7\tau)}$$

IV. Our final identity which follows also from the theory described here with the same proviso, but not immediately, is in a sense the companion to the identity in II in much the same way as the previous identity is the companion to the first identity given in I:

$$\sqrt{\frac{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,9\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)}}-\sqrt{\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(0,9\tau)}{\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)}}-\sqrt{\frac{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,9\tau)}{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)}}$$
$$=-3\sqrt{\frac{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,9\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,9\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,9\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)}}.$$

This identity is also equivalent to one of Ramanujan [B, p.352 (xi)]. The last two identities will be dealt with in a future publication.

References

[B] B. Berndt, *Ramanujan's Notebooks*, Part III, Springer-Verlag, New York, Heidelberg, Berlin, 1991.

[BB1] P. Borwien and J. Borwien, *A cubic counterpart of Jacobi's Identity and the AGM*, Trans. Amer. Math. Soc. **323** (1991), 691–701.

[BB2] P. Borwien and J. Borwien, Π *and the AGM*, Wiley, New York, 1987.

[FK1] H. M. Farkas and I. Kra, *Riemann Surfaces*, 2nd ed., Springer-Verlag, New York, Heidelberg, Berlin, 1992.

[FK2] H. M. Farkas and I. Kra, *Automorphic forms for subgroups of the modular group*, Israel J. Math. **82** (1993), 87–131.

[I] J. I. Igusa, *Theta Functions*, Springer-Verlag, New York, Heidelberg, Berlin, 1972.

[M] D. Mumford, *Tata Lectures on Theta I*, Birkhauser, Boston, Basel, Stuttgart, 1983.

[RF] H. E. Rauch and H. M. Farkas, *Theta Functions with Applications to Riemann Surfaces*, Williams and Wilkins, Baltimore, 1974.